

AMALGAMATIONS OF CLASSES OF BANACH SPACES WITH A MONOTONE BASIS

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ABSTRACT. It was proved by Argyros and Dodos that, for many classes \mathcal{C} of separable Banach spaces which share some property P , there exists an isomorphically universal space that satisfies P as well. We introduce a variant of their amalgamation technique which provides an isometrically universal space in the case that \mathcal{C} consists of spaces with a monotone Schauder basis. For example, we prove that if \mathcal{C} is a set of separable Banach spaces which is analytic with respect to the Effros-Borel structure and every $X \in \mathcal{C}$ is reflexive and has a monotone Schauder basis, then there exists a separable reflexive Banach space that is isometrically universal for \mathcal{C} .

1. INTRODUCTION AND THE MAIN RESULT

Let \mathcal{C} be a class of Banach spaces. We say that a Banach space X is *isomorphically (isometrically) universal for \mathcal{C}* if it contains an isomorphic (isometric) copy of every member of \mathcal{C} .

The present paper deals with universality questions in separable Banach space theory. Our aim is to find an isometric version of the amalgamation theory of S. A. Argyros and P. Dodos [1] and provide a method how to construct small isometrically universal spaces for small families of Banach spaces. Many of the results considered in the paper employs methods from descriptive set theory. The connection of universality problems and descriptive set theory, discovered by J. Bourgain [4, 5], deepened the theory and enabled several intrinsic questions to be understood. (See also [3], [11], [9], [13], for an introduction, see [18]).

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In 1968, W. Szlenk [28] proved that the class of separable reflexive spaces has no isomorphically universal element. (It had been shown some time ago by J. Lindenstrauss [25] that it has no isometrically universal element). He proved that a Banach space which is isomorphically universal for separable reflexive spaces has non-separable dual. His proof led to the famous Szlenk index which will be useful also in proofs of the present results.

Later, J. Bourgain [4] proved that, if a separable Banach space is isomorphically universal for separable reflexive spaces, then it is actually isomorphically universal for all separable Banach spaces. A somewhat different proof of this result was provided by B. Bossard [3] who showed that, if an analytic set of separable Banach spaces contains all separable reflexive spaces up to isomorphism, then it contains a space which is isomorphically universal for all separable Banach spaces. (An analytic set of Banach spaces is defined in Section 2). For a separable Banach space X , the set of all Banach spaces with an isomorphic copy in X is analytic. Therefore, Bourgain's result follows from Bossard's one.

Bossard's approach consists in constructing a tree space such that every infinite branch supports a universal space and every tree without infinite branches supports a reflexive space. It is possible to apply this approach on analogous questions concerning isometry as well. It was shown in [20] that, if a separable Banach space is isometrically universal for separable strictly convex spaces, then it is actually isometrically universal for all separable Banach spaces. The same result holds for the class of reflexive spaces [24].

In a work of S. A. Argyros and P. Dodos [1], the concept of a tree space turned out to be a powerful tool also for constructing universal spaces (see also [10]). When a set of separable Banach spaces \mathcal{C} is simple (in the sense that \mathcal{C} is analytic and every member has a Schauder basis), then it is possible to find a tree space such that the spaces supported by infinite branches are isomorphic copies of all members of \mathcal{C} . If the tree space is constructed properly, properties of spaces from \mathcal{C} can be preserved.

Some results of the Argyros-Dodos amalgamation theory are resumed in the following theorem (by a basis we mean a Schauder basis).

Theorem 1.1 ([1]). *Let \mathcal{P} be one of the following classes of separable Banach spaces:*

- *the class of spaces with a shrinking basis,*
- *the class of reflexive spaces with a basis,*

- the class of spaces with a basis which are not isomorphically universal for all separable Banach spaces.

Let \mathcal{C} be an analytic set of spaces from \mathcal{P} . Then there exists a Banach space E which belongs to \mathcal{P} and which contains a complemented isomorphic copy of every member of \mathcal{C} .

The reliance on a basis was dropped soon in works of P. Dodos and V. Ferenczi [11] and P. Dodos [9]. They proved that Theorem 1.1 holds (without the property that the copies are complemented) also for the following classes:

- the class of spaces with separable dual [11],
- the class of separable reflexive spaces [11],
- the class of separable spaces which are not isomorphically universal for all separable Banach spaces [9].

In the present work, we study the problem whether these results have an isometric version (see also [19, Problem 9]). We establish an isometric variant of Theorem 1.1.

A basis x_1, x_2, \dots is said to be *monotone* if the associated partial sum operators $P_n : \sum_{k=1}^{\infty} a_k x_k \mapsto \sum_{k=1}^n a_k x_k$ satisfy $\|P_n\| \leq 1$.

Theorem 1.2. *Let \mathcal{P} be one of the following classes of separable Banach spaces:*

- the class of spaces with a monotone shrinking basis,
- the class of reflexive spaces with a monotone basis,
- the class of spaces with a monotone basis which are not isometrically universal for all separable Banach spaces,
- the class of strictly convex spaces with a monotone basis.

Let \mathcal{C} be an analytic set of spaces from \mathcal{P} . Then there exists a Banach space E which belongs to \mathcal{P} and which contains an 1-complemented isometric copy of every member of \mathcal{C} .

We do not know whether the reliance on a basis can be dropped, similarly as in the isomorphic setting. The requirement of the existence of a monotone basis is a weak point of Theorem 1.2, but it is possible that the theorem will be helpful for more powerful results to be obtained in future.

We include here several remarks concerning Theorem 1.2.

(I) For the class of spaces with a shrinking basis and the class of reflexive spaces with a basis, it is not difficult to show that Theorem 1.1 follows from Theorem 1.2.

(II) The theorem remains valid if we consider monotone finite dimensional decompositions instead of monotone bases. A variant of a

space constructed by S. Prus [26] can be provided. Since the class of super-reflexive spaces is analytic, there exists a separable reflexive space which contains an 1-complemented isometric copy of every super-reflexive space with a monotone finite dimensional decomposition. Since $F \oplus_2 \ell_2$ is super-reflexive for each finite dimensional space F , we obtain also the result of A. Szankowski [27] which states that there exists a separable reflexive space, isometrically universal for all finite dimensional spaces.

(III) It is possible to use the methods developed in the paper for the construction of a Pełczyński universal space which contains an 1-complemented isometric copy of every Banach space with a monotone basis (see Definition 9.3). Similar examples have been constructed by J. Garbulińska-Węgrzyn [14, 15].

(IV) Theorem 1.2 holds for more general classes than the class of non-universal spaces. Let Z be a separable Banach space for which there are an $a \in Z$ and a subset $H \subset Z$ whose closed linear span contains an isometric copy of Z and such that, for every $h \in H$, there is an $\varepsilon > 0$ with $\|a \pm \varepsilon h\| = \|a\|$. Then the theorem holds for the class of spaces with a monotone basis not containing an isometric copy of Z . Among the universal space $Z = C(\{0, 1\}^{\mathbb{N}})$, the required property is fulfilled e.g. by the spaces $Z = c_0$ and $Z = \ell_1$.

(V) If a separable Banach space X is isomorphically universal for separable Schur spaces, then it is actually isomorphically universal for all separable Banach spaces. This follows from methods in [3] (see [6, Corollary 51]). We are able to prove the isometric version of this statement (see Remark 3.7).

It is not known if the class of Schur spaces with a basis has the property from Theorem 1.1. It is not clear whether the tree space method can be used in this case. However, the property is fulfilled by the related class of ℓ_1 -saturated spaces with a basis (see [1, Theorem 91]).

2. PRELIMINARIES

By $\Lambda^{<\mathbb{N}}$ we denote the set of all finite sequences of elements of a set Λ , including the empty sequence \emptyset . That is,

$$\Lambda^{<\mathbb{N}} = \bigcup_{l=0}^{\infty} \Lambda^l$$

where $\Lambda^0 = \{\emptyset\}$. The length of an $\eta \in \Lambda^{<\mathbb{N}}$ is denoted by $|\eta|$. If $\eta \in \Lambda^{<\mathbb{N}}$ and $\nu \in \Lambda^{<\mathbb{N}} \cup \Lambda^{\mathbb{N}}$, then by $\eta \subset \nu$ we mean that η is an initial segment of ν , i.e., the length of η is less than or equal to the

length of ν and $\eta(i) = \nu(i)$ for $1 \leq i \leq |\eta|$. By $(n_1, \dots, n_k)^\wedge n$ we mean (n_1, \dots, n_k, n) . A subset T of $\Lambda^{<\mathbb{N}}$ is called a *tree on Λ* if

$$\eta \subset \nu \text{ \& } \nu \in T \quad \Rightarrow \quad \eta \in T.$$

Moreover, a set $T \subset \Lambda^{<\mathbb{N}} \setminus \{\emptyset\}$ is called an *unrooted tree on Λ* if $T \cup \{\emptyset\}$ is a tree on Λ . An (unrooted) tree T is called *pruned* if every $\eta \in T$ has a proper extension $\nu \supset \eta, \nu \neq \eta, \nu \in T$. The set of all infinite branches of T , i.e., sequences $\nu \in \Lambda^{\mathbb{N}}$ such that T contains all non-empty initial segments of ν , is denoted by $[T]$. An (unrooted) tree T is called *well-founded* if it does not have an infinite branch.

A *Polish space (topology)* means a separable completely metrizable space (topology). A set P equipped with a σ -algebra is called a *standard Borel space* if the σ -algebra is generated by a Polish topology on P . A subset of a standard Borel space is called *analytic* if it is a Borel image of a Polish space.

The following lemma can be found e.g. in [22, (25.2)].

Lemma 2.1. *A subset $A \subset \mathbb{N}^{\mathbb{N}}$ is analytic if and only if there is a pruned tree T on $\mathbb{N} \times \mathbb{N}$ such that $A = p[T]$ where $p : \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$ denotes the projection on the first coordinate.*

For a topological space X , the set $\mathcal{F}(X)$ of all closed subsets of X is equipped with the *Effros-Borel structure*, defined as the σ -algebra generated by the sets

$$\{F \in \mathcal{F}(X) : F \cap U \neq \emptyset\}$$

where U varies over open subsets of X . If X is Polish, then, equipped with this σ -algebra, $\mathcal{F}(X)$ forms a standard Borel space.

The *standard Borel space of separable Banach spaces* is defined by

$$\mathcal{SE}(C([0, 1])) = \{F \in \mathcal{F}(C([0, 1])) : F \text{ is linear}\}$$

and considered as a subspace of $\mathcal{F}(C([0, 1]))$.

For a separable Banach space X and an $F \subset B_{X^*}$, let

$$F'_\varepsilon = F \setminus \bigcup \{U \subset X^* : U \text{ is } w^*\text{-open, } \text{diam}(U \cap F) < \varepsilon\}, \quad \varepsilon > 0,$$

and recursively

$$F_\varepsilon^{(0)} = F, \quad F_\varepsilon^{(\alpha)} = \bigcap_{\beta < \alpha} (F_\varepsilon^{(\beta)})'_\varepsilon, \quad \varepsilon > 0.$$

We define

$$\text{Sz}_\varepsilon(F) = \min(\{\omega_1\} \cup \{\alpha < \omega_1 : F_\varepsilon^{(\alpha)} = \emptyset\}), \quad \varepsilon > 0,$$

$$\text{Sz}(F) = \sup\{\text{Sz}_\varepsilon(F) : \varepsilon > 0\}.$$

The *Szlenk index* of X is defined by $\text{Sz}(X) = \text{Sz}(B_{X^*})$.

For an (unrooted) tree T and a system $\{x_\eta : \eta \in T\}$ of elements of a Banach space, we define

$$\sum_{\eta \in T} x_\eta = \lim_{S \rightarrow T} \sum_{\eta \in S} x_\eta \quad (\text{if the limit exists})$$

where the limit is taken over all finite subtrees $S \subset T$ directed by inclusion.

The notions and notation we use but do not introduce here are classical and well explained e.g. in [12] and [22].

3. THE INITIAL TREE SPACE CONSTRUCTION

In this section, we introduce our basic tool for constructing tree spaces. Basically, two ways have been developed how to extract the norm of a tree space from the norms of the subspaces supported by infinite branches (excluding the norm constructed in [24]). The first way, based on the well known James tree space [21], was employed mainly in works of B. Bossard [3] and S. A. Argyros and P. Dodos [1].

However, we follow the second way which is more suitable for isometric problems. The method was introduced by B. Bossard [2] and employed later by G. Godefroy [17] and G. Godefroy and N. J. Kalton [20]. In fact, the tree space from the following definition is a simplified version of the original tree space from [2] which will be introduced later in Definition 5.1 nevertheless.

Definition 3.1. Let Λ be a countable set and let T be a pruned unrooted tree on Λ . For every $\sigma \in [T]$, let $(F_\sigma, \|\cdot\|_\sigma)$ be a Banach space with a monotone basis $f_1^\sigma, f_2^\sigma, \dots$ and let these bases have the property that $f_1^\sigma, f_2^\sigma, \dots, f_l^\sigma$ and $f_1^\varphi, f_2^\varphi, \dots, f_l^\varphi$ are 1-equivalent whenever σ and φ have the same initial segment of length l .

Let us consider the norm on $c_{00}(T)$ defined by

$$(1) \quad \|x\| = \sup_{\sigma \in [T]} \left\| \sum_{\eta \subset \sigma} x(\eta) f_{|\eta|}^\sigma \right\|_\sigma$$

and, for every unrooted subtree $S \subset T$, the projection

$$(2) \quad P_S x = \mathbf{1}_S \cdot x.$$

From the monotonicity of the bases f_n^σ , we obtain

$$(3) \quad \|P_S x\| \leq \|x\|.$$

Finally, we define E as a completion of $(c_{00}(T), \|\cdot\|)$. The members of the canonical basis of $c_{00}(T)$ will be denoted by e_η (i.e., $e_\eta = \mathbf{1}_{\{\eta\}}$). We note that the system $\{e_\eta : \eta \in T\}$ is a basis of E , which follows

from the observation that the property $x = \lim_{S \rightarrow T} P_S x$ extends from $c_{00}(T)$ to its closure E , due to the uniform boundedness of the projections P_S . The basis is monotone in the sense of formula (3).

Since $\{e_\eta : \eta \in T\}$ is a basis of E , we are allowed to consider all elements of E as systems $x = \{x(\eta)\}_{\eta \in T}$ of scalars. In this way, formulae (1), (2) and (3) remain valid for every $x \in E$. We will denote the members of the corresponding dual system by e_η^* (i.e., $e_\eta^*(x) = x(\eta)$).

For every $\sigma \in [T]$, we further define spaces

$$(4) \quad \begin{aligned} E_\sigma &= \{x \in E : \eta \not\subset \sigma \Rightarrow x(\eta) = 0\}, \\ E_\sigma^* &= \{x^* \in E^* : \eta \not\subset \sigma \Rightarrow x^*(e_\eta) = 0\} \end{aligned}$$

and a projection

$$(5) \quad P_\sigma = P_{\{(\sigma_1), (\sigma_1, \sigma_2), \dots\}}.$$

We also denote

$$(6) \quad \Phi = \bigcup_{\sigma \in [T]} B_{E_\sigma} \quad \text{and} \quad \Psi = \bigcup_{\sigma \in [T]} B_{E_\sigma^*}.$$

Fact 3.2. *For every $\sigma \in [T]$, the basis $f_1^\sigma, f_2^\sigma, \dots$ of F_σ is 1-equivalent with the basis $e_{(\sigma_1)}, e_{(\sigma_1, \sigma_2)}, \dots$ of E_σ . In particular, the space E contains an 1-complemented isometric copy of F_σ for every $\sigma \in [T]$.*

Proof. Let $f = \sum_{n=1}^\infty r_n f_n^\sigma$ and $x = \sum_{n=1}^\infty r_n e_{(\sigma_1, \dots, \sigma_n)}$ where $r_n \neq 0$ for finitely many indices n only. We have

$$\left\| \sum_{\eta \subset \sigma} x(\eta) f_{|\eta|}^\sigma \right\|_\sigma = \left\| \sum_{n=1}^\infty r_n f_n^\sigma \right\|_\sigma = \|f\|_\sigma,$$

and so it remains to check that

$$\left\| \sum_{\nu \subset \tau} x(\nu) f_{|\nu|}^\tau \right\|_\tau \leq \|f\|_\sigma$$

for each $\tau \in [T] \setminus \{\sigma\}$. Let η be the longest segment such that $\eta \subset \sigma$ and $\eta \subset \tau$, and let l be its length. Then

$$\left\| \sum_{\nu \subset \tau} x(\nu) f_{|\nu|}^\tau \right\|_\tau = \left\| \sum_{n=1}^l r_n f_n^\tau \right\|_\tau = \left\| \sum_{n=1}^l r_n f_n^\sigma \right\|_\sigma \leq \|f\|_\sigma.$$

The second part of the assertion follows from $E_\sigma = P_\sigma E$. □

Fact 3.3. *For $x \in E$, we have*

$$\|P_\sigma x\| = \sup_{x^* \in B_{E_\sigma^*}} |x^*(x)|.$$

For $x^* \in E^*$, we have

$$\|P_\sigma^* x^*\| = \sup_{x \in B_{E_\sigma}} |x^*(x)|.$$

Proof. The fact follows directly from the observation that $P_\sigma B_E = B_{E_\sigma}$ and $P_\sigma^* B_{E^*} = B_{E_\sigma^*}$. \square

Lemma 3.4. *The set Ψ is compact in the weak* topology of E^* and its convex hull is w^* -dense in B_{E^*} .*

Proof. To show that Ψ is w^* -compact, we just write

$$\Psi = B_{E^*} \setminus \bigcup \left\{ x^* \in E^* : x^*(e_\eta) \neq 0 \text{ \& } x^*(e_\nu) \neq 0 \right\}$$

where the union is taken over all couples η, ν of incomparable segments in T . Using (1) in combination with Facts 3.2 and 3.3, we obtain for $x \in E$ that

$$\|x\| = \sup_{\sigma \in [T]} \|P_\sigma x\| = \sup_{\sigma \in [T]} \sup_{x^* \in B_{E_\sigma^*}} |x^*(x)| = \sup_{x^* \in \Psi} |x^*(x)|.$$

Now, to prove that the convex hull of Ψ is w^* -dense in B_{E^*} , it is sufficient to apply the Hahn-Banach theorem. \square

Proposition 3.5. *If the basis $f_1^\sigma, f_2^\sigma, \dots$ is shrinking for every $\sigma \in [T]$, then the basis $\{e_\eta : \eta \in T\}$ is also shrinking.*

Proof. Let us fix an increasing sequence T_1, T_2, \dots of finite unrooted trees with $\bigcup_{n=1}^\infty T_n = T$. We show first that

$$x^* \in \Psi \quad \Rightarrow \quad P_{T_n}^* x^* \rightarrow x^*.$$

Given a $\sigma \in [T]$, we check the implication for the elements of $B_{E_\sigma^*}$. By Fact 3.2, the sequence $e_{(\sigma_1)}, e_{(\sigma_1, \sigma_2)}, \dots$ is a shrinking basis of E_σ . By Fact 3.3, the elements of E_σ^* satisfy

$$\|x^*\| = \sup_{x \in B_{E_\sigma}} |x^*(x)|, \quad x^* \in E_\sigma^*.$$

Hence E_σ^* is (isometric to) the dual of E_σ indeed. The dual sequence $e_{(\sigma_1)}^*, e_{(\sigma_1, \sigma_2)}^*, \dots$ is a basis of E_σ^* . It follows that $P_{T_n}^* x^* \rightarrow x^*$ for each $x^* \in E_\sigma^*$.

Now, let $y^* \in B_{E^*}$. By Lemma 3.4, there exists a probability measure μ on Ψ such that

$$y^* = \int_\Psi x^* d\mu(x^*).$$

Therefore,

$$\begin{aligned} \lim_{n \rightarrow \infty} \|y^* - P_{T_n}^* y^*\| &= \lim_{n \rightarrow \infty} \left\| \int_{\Psi} (x^* - P_{T_n}^* x^*) d\mu(x^*) \right\| \\ &\leq \lim_{n \rightarrow \infty} \int_{\Psi} \|x^* - P_{T_n}^* x^*\| d\mu(x^*) \\ &= \int_{\Psi} \lim_{n \rightarrow \infty} \|x^* - P_{T_n}^* x^*\| d\mu(x^*) = \int_{\Psi} 0 d\mu(x^*) = 0. \end{aligned}$$

This proves that y^* belongs to the closed linear span of the functionals $e_{\eta}^*, \eta \in T$. \square

Lemma 3.6. *If the space F_{σ} is reflexive for every $\sigma \in [T]$, then the set Φ is compact in the weak topology of E .*

Proof. Let x_1, x_2, \dots be a sequence in Φ . We want to find a subsequence x_{n_k} which converges weakly to an $x \in \Phi$. By Proposition 3.5, it is sufficient to check that

$$x_{n_k}(\eta) \rightarrow x(\eta), \quad \eta \in T.$$

Using a diagonal argument, we choose the subsequence x_{n_k} so that

$$x_{n_k}(\eta) \rightarrow u(\eta), \quad \eta \in T,$$

for a system of scalars $u = \{u(\eta)\}_{\eta \in T}$. It remains to show that this system forms the coordinates of an $x \in \Phi$.

First, we realize that there is a $\sigma \in [T]$ such that u is supported by the branch $\{(\sigma_1), (\sigma_1, \sigma_2), \dots\}$. Indeed, if $u(\eta) \neq 0 \neq u(\nu)$ for some incomparable $\eta, \nu \in T$, then $x_{n_k}(\eta) \neq 0 \neq x_{n_k}(\nu)$ for a large enough k , which is not allowed by the definition of Φ .

By Fact 3.2 and our assumption, E_{σ} is reflexive. A subsequence of $P_{\sigma} x_{n_k}$ converges weakly to an $x \in B_{E_{\sigma}}$, and this limit satisfies $x(\eta) = u(\eta)$ for every $\eta \in T$. \square

Remark 3.7. (a) If $S \subset T$ is a well-founded unrooted subtree, then the subspace

$$(7) \quad H(S) = \overline{\text{span}}\{e_{\eta}^* : \eta \in S\}$$

of E^* has the Schur property. Let us assume that $H(S)$ is not Schur and denote $H_{\nu}(S) = \overline{\text{span}}\{e_{\eta}^* : \eta \in S \text{ and } \nu \subset \eta\}$ for $\nu \in T \cup \{\emptyset\}$. It is sufficient to prove that

$$H_{\nu}(S) \text{ is not Schur} \quad \Rightarrow \quad H_{\nu \wedge n}(S) \text{ is not Schur for some } n,$$

as this allows us to find an infinite branch of S . One can show that

$$\left\| \sum_{n=1}^m x_n^* \right\| \geq \frac{1}{2} \sum_{n=1}^m \|x_n^*\|, \quad x_n^* \in H_{\nu \wedge n}(S), \quad n = 1, 2, \dots, m.$$

Therefore, a hyperplane of $H_\nu(S)$ (or $H_\nu(S)$ itself when $\nu = \emptyset$) is isomorphic to the ℓ_1 -sum of $H_{\nu \wedge 1}(S), H_{\nu \wedge 2}(S), \dots$, and the implication follows.

(b) If a separable Banach space X contains an isometric copy of every separable Schur space, then it contains an isometric copy of every separable Banach space. To show this, we follow a method of B. Bossard [3]. Let x_1, x_2, \dots be a monotone basis of $C([0, 1])$ (see e.g. [8, p. 34]) and f_1, f_2, \dots be the dual basic sequence in $C([0, 1])^*$. Let $T = \mathbb{N}^{<\mathbb{N}} \setminus \{\emptyset\}$, $f_n^\sigma = f_n$ and $F_\sigma = \overline{\text{span}}\{f_n : n \in \mathbb{N}\}$ for every $\sigma \in \mathbb{N}^\mathbb{N}$. In this setting, let $H(S)$ be given by (7). Let Tr be the subspace of 2^T consisting of all unrooted trees on \mathbb{N} and let WF be the set of all well-founded $S \in \text{Tr}$. Consider the set

$$\mathcal{A} = \{S \in \text{Tr} : X \text{ contains an isometric copy of } H(S)\}.$$

Then \mathcal{A} is analytic (see [17, Lemma 7 and Lemma 8]) and it contains WF , due to our assumption. Since WF is not analytic (see e.g. [22, (27.1)]), there is an $S \in \mathcal{A} \setminus \text{WF}$. So, X contains an isometric copy of $H(S)$ for some $S \notin \text{WF}$, which contains an isometric copy of $C([0, 1])$.

(c) Assume that, for every $\sigma \in [T]$, the space $\overline{\text{span}}\{f_n^* : n \in \mathbb{N}\}$ has the Schur property, where f_1^*, f_2^*, \dots is the dual basic sequence of the basis $f_1^\sigma, f_2^\sigma, \dots$. We do not know whether $H = \overline{\text{span}}\{e_\eta^* : \eta \in T\}$ has necessarily the Schur property in such a case. It is possible to show that $B_H = \overline{\text{co}}(H \cap \Psi)$ and that every weakly convergent sequence in $H \cap \Psi$ is convergent, but this does not seem to be sufficient for H to be Schur.

4. THE INTERPOLATION METHOD

The aim of this section is to provide a reflexive variant of the tree space from Definition 3.1. As well as the authors of [1], we apply the Davis-Figiel-Johnson-Pełczyński interpolation method.

Definition 4.1 ([7]). Let W be a bounded, closed, convex and symmetric subset of a Banach space X . For each $n \in \mathbb{N}$, let $\|\cdot\|_n$ be the equivalent norm given by

$$B_{(X, \|\cdot\|_n)} = \overline{2^n W + 2^{-n} B_X}.$$

The *2-interpolation space of the pair (X, W)* is defined as the space $(Y, |||\cdot|||)$ where

$$|||x||| = \left(\sum_{n=1}^{\infty} \|x\|_n^2 \right)^{1/2}, \quad x \in X,$$

and

$$Y = \{x \in X : |||x||| < \infty\}.$$

Lemma 4.2. *Let $P : X \rightarrow X$ be a projection such that $\|P\| \leq 1$ and $PW \subset W$. Then we have*

$$|||Px||| \leq |||x|||, \quad x \in X.$$

If, moreover, $PW = PB_X$, then there is a constant $c > 0$ such that

$$|||x||| = c\|x\|, \quad x \in PX.$$

In particular, Y contains an 1-complemented isometric copy of PX .

Proof. The inequality $|||Px||| \leq |||x|||$ (which can be proven quite easily actually) follows from [7, p. 316, Lemma 1(viii)]. To provide a suitable constant $c > 0$, it is sufficient to show that

$$\|x\|_n = \frac{1}{2^n + 2^{-n}} \|x\|, \quad x \in PX.$$

Let $x \in PX$ be given. We will assume that $\|x\| = 1$. Since $x \in B_X$ and $x = Px \in PB_X = PW \subset W$, we have $(2^n + 2^{-n})x \in B_{(X, \|\cdot\|_n)}$. Therefore, $(2^n + 2^{-n})\|x\|_n \leq 1 = \|x\|$.

Let $0 < \theta < 1/\|x\|_n$ be chosen arbitrarily. We have $\|\theta x\|_n < 1$, and so θx belongs to $2^n W + 2^{-n} B_X$. We obtain $\theta x = 2^n w + 2^{-n} y$ for some $w \in W$ and $y \in B_X$. Since $Pw \in PW = PB_X \subset B_X$, we have $\theta x = \theta Px = 2^n Pw + 2^{-n} Py$ and $\theta\|x\| \leq 2^n + 2^{-n}$. Due to the choice of θ , the inequality $\|x\| \leq (2^n + 2^{-n})\|x\|_n$ follows. \square

Definition 4.3. Adopting the notation from Definition 3.1, we define A as the 2-interpolation space of the pair $(E, \overline{\text{co}} \Phi)$.

Fact 4.4. *The system $\{e_\eta : \eta \in T\}$ is a monotone basis of A .*

Proof. The associated projections P_S satisfy $|||P_S x||| \leq |||x|||$ by (3) and Lemma 4.2. The fact thus follows from [7, p. 316, Lemma 1(ix)]. \square

Fact 4.5. *A contains an 1-complemented isometric copy of F_σ for every $\sigma \in [T]$.*

Proof. Recall that F_σ is isometric to $E_\sigma = P_\sigma E$ by Fact 3.2. The assumptions of Lemma 4.2 are met for $P = P_\sigma$, since $P_\sigma(\overline{\text{co}} \Phi) \subset \overline{\text{co}} \Phi$ and $P_\sigma(\overline{\text{co}} \Phi) = B_{E_\sigma} = P_\sigma B_E$. \square

Proposition 4.6. *If the space F_σ is reflexive for every $\sigma \in [T]$, then the space A is also reflexive.*

Proof. By Lemma 3.6 and the Krein-Smulian theorem, the set $\overline{\text{co}} \Phi$ is weakly compact. To show that A is reflexive, it is sufficient to apply [7, p. 313, Lemma 1(iv)]. \square

5. A ROTUND VERSION OF THE TREE SPACE

The following definition of a tree space is based on a construction from [2] which was applied also in [17] and [20]. Regarding the results from these papers, it is not surprising that this tree space preserves strict convexity of the norm. However, it turns out that the method is suitable also for amalgamating spaces which are not isometrically universal (see Proposition 5.5).

Definition 5.1. Let $\Lambda, T, (F_\sigma, \|\cdot\|_\sigma)$ and f_n^σ be as in Definition 3.1. Suppose moreover that there are positive constants c_1, c_2, \dots such that, for every $\sigma \in [T]$,

$$(8) \quad \|\pi_n f\|_\sigma^2 \geq \|\pi_{n-1} f\|_\sigma^2 + c_n^2 |f_n^*(f)|^2, \quad f \in F_\sigma, n \in \mathbb{N},$$

where f_1^*, f_2^*, \dots is the dual basic sequence and π_0, π_1, \dots is the sequence of partial sum operators associated with the basis $f_1^\sigma, f_2^\sigma, \dots$.

For every $x \in c_{00}(T)$, let us consider the formulae

$$(9) \quad |||x|||_\sigma^2 = \left\| \sum_{\eta \subset \sigma} x(\eta) f_{|\eta|}^\sigma \right\|_\sigma^2 + \sum_{\eta \not\subset \sigma} c_{|\eta|}^2 |x(\eta)|^2, \quad \sigma \in [T],$$

$$(10) \quad |||x||| = \sup_{\sigma \in [T]} |||x|||_\sigma$$

and

$$(11) \quad P_S x = \mathbf{1}_S \cdot x,$$

where $S \subset T$ is an unrooted subtree. From the monotonicity of the bases f_n^σ , we obtain

$$(12) \quad |||P_S x||| \leq |||x|||.$$

Finally, we define B as a completion of $(c_{00}(T), |||\cdot|||)$. Again, the system $\{b_\eta = \mathbf{1}_{\{\eta\}} : \eta \in T\}$ is a basis of B which is monotone in the sense of formula (12). Therefore, we are allowed to consider all elements of B as systems $x = \{x(\eta)\}_{\eta \in T}$ of scalars. In this way, formulae (9), (10), (11) and (12) remain valid for every $x \in B$.

For every $\sigma \in [T]$, we further denote

$$(13) \quad B_\sigma = \{x \in B : \eta \not\subset \sigma \Rightarrow x(\eta) = 0\},$$

$$(14) \quad P_\sigma = P_{\{(\sigma_1), (\sigma_1, \sigma_2), \dots\}}.$$

Fact 5.2. For every $\sigma \in [T]$, the basis $f_1^\sigma, f_2^\sigma, \dots$ of F_σ is 1-equivalent with the basis $b_{(\sigma_1)}, b_{(\sigma_1, \sigma_2)}, \dots$ of B_σ . In particular, the space B contains an 1-complemented isometric copy of F_σ for every $\sigma \in [T]$.

We do not prove the fact, as an analogous statement appeared in [17] and [20]. Actually, the fact can be proven similarly as Fact 3.2, with the difference that (8) is applied. The proof of the following lemma, which is essentially contained in [20, p. 186], is also skipped.

Lemma 5.3 ([20]). *For every $x \in B$, the supremum in (10) is attained.*

Lemma 5.4. *Let $[u, v]$ be a non-degenerate line segment in B such that $|||\cdot|||$ is constant on $[u, v]$. Let $w = \frac{1}{2}(u + v)$ and the supremum in (10) for $x = w$ be attained at a $\sigma \in [T]$. Then $v - u \in B_\sigma$ and $[P_\sigma u, P_\sigma v]$ is also a non-degenerate line segment on which $|||\cdot|||$ is constant.*

Proof. Let us consider a seminorm

$$|x|_\sigma^2 = \sum_{\eta \notin \sigma} c_{|\eta|}^2 |x(\eta)|^2, \quad x \in B.$$

Using Fact 5.2, we obtain

$$|||x|||_\sigma^2 = |||P_\sigma x|||^2 + |x|_\sigma^2, \quad x \in B.$$

We can compute

$$|||w||| = |||w|||_\sigma \leq \frac{1}{2}(|||u|||_\sigma + |||v|||_\sigma) \leq \frac{1}{2}(|||u||| + |||v|||) = |||w|||,$$

and it is clear that all these norms must be equal. Thus,

$$\begin{aligned} 0 &= 2|||u|||_\sigma^2 + 2|||v|||_\sigma^2 - 4|||w|||_\sigma^2 \\ &= 2|||P_\sigma u|||^2 + 2|||P_\sigma v|||^2 - 4|||P_\sigma w|||^2 + 2|u|_\sigma^2 + 2|v|_\sigma^2 - 4|w|_\sigma^2 \\ &= (|||P_\sigma u||| - |||P_\sigma v|||)^2 + (|||P_\sigma u||| + |||P_\sigma v|||)^2 - |||P_\sigma(u + v)|||^2 \\ &\quad + (|u|_\sigma - |v|_\sigma)^2 + (|u|_\sigma + |v|_\sigma)^2 - |u + v|_\sigma^2. \end{aligned}$$

It follows that

$$(15) \quad |||P_\sigma u||| = |||P_\sigma v|||, \quad |||P_\sigma(u + v)||| = |||P_\sigma u||| + |||P_\sigma v|||,$$

$$(16) \quad |u|_\sigma = |v|_\sigma, \quad |u + v|_\sigma = |u|_\sigma + |v|_\sigma.$$

By (15), the norm $|||\cdot|||$ is constant on $[P_\sigma u, P_\sigma v]$. By (16), the points u and v satisfy $u(\eta) = v(\eta)$ for every $\eta \notin \sigma$. That is, $u - P_\sigma u = v - P_\sigma v$. Therefore, $v - u = P_\sigma v - P_\sigma u \in B_\sigma$ and the segment $[P_\sigma u, P_\sigma v]$ is non-degenerate. \square

Proposition 5.5. (a) *If none $F_\sigma, \sigma \in [T]$, is isometrically universal for all separable Banach spaces, then B is also non-universal.*

(b) *If every $F_\sigma, \sigma \in [T]$, is strictly convex, then B is strictly convex.*

Proof. (a) Assume that B is isometrically universal for all separable Banach spaces. Let us denote

$$\Delta = \{0, 1\}^{\mathbb{N}}, \quad \Delta(i) = \{\gamma \in \Delta : \gamma(1) = i\}, \quad i = 0, 1,$$

$$Z = C(\Delta), \quad Z(i) = \{h \in Z : \gamma \notin \Delta(i) \Rightarrow h(\gamma) = 0\}, \quad i = 0, 1.$$

Considering an isometry $I : Z \rightarrow B$, we denote

$$x = I(\mathbf{1}_{\Delta(0)}).$$

Lemma 5.3 provides us with a $\sigma \in [T]$ at which the supremum in (10) is attained. We claim that the space B_σ (and therefore the space F_σ by Fact 5.2) is universal, showing that I maps $Z(1)$ into B_σ .

Given an $h \in Z(1)$ with $\|h\| \leq 1$, we observe that $\|\mathbf{1}_{\Delta(0)}\| = \|\mathbf{1}_{\Delta(0)} \pm h\| = 1$, and so $\|x\| = \|x \pm Ih\| = 1$. By Lemma 5.4, we have $Ih \in B_\sigma$.

(b) Assume that B is not strictly convex. It means that $\|\cdot\|$ is constant on a non-degenerate line segment $[u, v]$. Let $x = \frac{1}{2}(u + v)$ and let the supremum in (10) be attained at a $\sigma \in [T]$ (which is provided by Lemma 5.3). By Lemma 5.4, the space B_σ is not strictly convex. Since B_σ and F_σ are isometric (see Fact 5.2), the proof is finished. \square

6. CONSTRUCTION OF BRANCHES

In the isomorphic setting, it is possible to construct a tree space in a way such that isomorphic copies of the spaces we want to amalgamate are placed on the infinite branches (as mentioned in the introduction). In the isometric setting, we are not allowed to renorm the spaces, and an additional embedding result is demanded.

We prove that a Banach space X with a monotone basis can be embedded into another (not much bigger) Banach space F with a monotone basis f_1, f_2, \dots such that the subspaces $\text{span}\{f_1, \dots, f_d\}$ are chosen from a countable family of spaces. To this purpose, we employ the following notion which was useful also in [14, 15, 16, 23].

Definition 6.1. A Banach space Z is called *rational* if $Z = \mathbb{R}^d$ with a norm such that its unit ball is generated by finitely many points whose all coordinates are rational numbers.

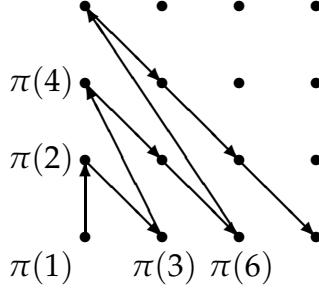
We note that the spaces which have a basis consisting of d elements will be often identified with \mathbb{R}^d in the obvious way.

The main goal of this section is to prove the following result. Its proof is based on a construction provided in [24, Section 4] (which was based on a construction from [20] in turn) but the present method is considerably simpler.

Proposition 6.2. *Let X be a Banach space and e_1, e_2, \dots be a monotone basis of X . Then there exists a Banach space F with a monotone basis f_1, f_2, \dots such that:*

- (1) F is isomorphic to $\ell_2(X)$.
- (2) If the basis e_1, e_2, \dots is shrinking, then the basis f_1, f_2, \dots is shrinking.
- (3) For every $d \in \mathbb{N}$, the space $\text{span}\{f_1, f_2, \dots, f_d\}$, identified with \mathbb{R}^d , is rational.
- (4) F contains an 1-complemented isometric copy of X .

Definition 6.3. By π we will denote the bijection $\mathbb{N} \rightarrow \mathbb{N}^2$ given by $\pi(1) = (1, 1), \pi(2) = (1, 2), \pi(3) = (2, 1), \pi(4) = (1, 3)$ etc.



Definition 6.4. For every $d \in \mathbb{N}$, let us fix an ordering of all monotone rational norms on \mathbb{R}^d into a sequence $|\cdot|_{d,1}, |\cdot|_{d,2}, \dots$.

Let e_1, e_2, \dots be a normalized monotone basis of a Banach space $(X, \|\cdot\|_X)$. Let $f_i = e_{\pi(i)}$ where $e_{(n,k)}$ stands for the element of $\ell_2(X)$ which has e_k on the n -th place and 0 elsewhere. Let us moreover denote

$$(17) \quad F_d = \text{span}\{f_1, f_2, \dots, f_d\}.$$

For every $d \in \mathbb{N}$, let $l_d = l_d(e_1, e_2, \dots)$ be the least natural number such that the monotone rational norm $|\cdot|_d = |\cdot|_{d,l_d}$ satisfies

$$(18) \quad \left(1 - \frac{1}{2^{2d+1}}\right) \|f\|_{\ell_2(X)} \leq |f|_d \leq \left(1 - \frac{1}{2^{2d+2}}\right) \|f\|_{\ell_2(X)}, \quad f \in F_d.$$

The formula is valid for some monotone rational norm on F_d , as the sequence f_1, f_2, \dots is a monotone basis of $\ell_2(X)$.

We define a space $F = F(e_1, e_2, \dots)$ with a norm $\|\cdot\|$ by

$$(19) \quad B_{(F, \|\cdot\|)} = \overline{\text{co}} \bigcup_{d=1}^{\infty} B_{(F_d, |\cdot|_d)}.$$

We also define operators

$$(20) \quad T : \ell_2(X) \rightarrow X, \quad (x_1, x_2, \dots) \mapsto \frac{\sqrt{3}}{2} \left(x_1 + \frac{1}{2}x_2 + \frac{1}{4}x_3 + \dots \right),$$

$$(21) \quad U : X \rightarrow \ell_2(X), \quad x \mapsto \frac{\sqrt{3}}{2} \left(x, \frac{1}{2}x, \frac{1}{4}x, \dots \right).$$

The sequence of partial sum operators associated with the basis f_1, f_2, \dots will be denoted by P_1, P_2, \dots .

Lemma 6.5. *We have $F = \ell_2(X)$ and the norm $\|\cdot\|$ fulfills*

$$(22) \quad \frac{7}{8}\|f\|_{\ell_2(X)} \leq \|f\| \leq \|f\|_{\ell_2(X)}, \quad f \in F.$$

The basis f_1, f_2, \dots forms a monotone basis of $(F, \|\cdot\|)$ which is shrinking if the basis e_1, e_2, \dots is shrinking. Finally, for every $d \in \mathbb{N}$, we have

$$(23) \quad B_{(F_d, \|\cdot\|)} = \text{co} \bigcup_{j=1}^d B_{(F_j, |\cdot|_j)}.$$

In particular, the space $(F_d, \|\cdot\|)$ is rational.

Proof. By (18), we have $\frac{7}{8}\|f\|_{\ell_2(X)} \leq |f|_d \leq \|f\|_{\ell_2(X)}$ for $f \in F_d$. Thus,

$$B_{(F_d, |\cdot|_d)} \subset \frac{8}{7}B_{\ell_2(X)} \quad \text{and} \quad B_{\ell_2(X)} \cap F_d \subset B_{(F_d, |\cdot|_d)} \subset B_{(F, \|\cdot\|)},$$

and it follows that

$$B_{(F, \|\cdot\|)} \subset \frac{8}{7}B_{\ell_2(X)} \quad \text{and} \quad B_{\ell_2(X)} \subset B_{(F, \|\cdot\|)}.$$

Clearly, if the basis e_1, e_2, \dots is shrinking, then the basis f_1, f_2, \dots is shrinking. To show that it is monotone with respect to $\|\cdot\|$, it is sufficient to realize that the associated partial sum operators P_1, P_2, \dots map the unit ball of $(F_d, |\cdot|_d)$ into itself, and consequently that the unit ball of $(F, \|\cdot\|)$ has the same property. To show (23), it is sufficient to prove that P_d maps the unit ball of $(F_j, |\cdot|_j)$, where $j > d$, into the unit ball of $(F_d, |\cdot|_d)$. For $f \in F_j$, we can compute

$$|P_d f|_d \leq \left(1 - \frac{1}{2^{2d+2}}\right) \|P_d f\|_{\ell_2(X)} \leq \left(1 - \frac{1}{2^{2j+1}}\right) \|f\|_{\ell_2(X)} \leq |f|_j,$$

which completes the proof. \square

Lemma 6.6. *We have $\|Tf\|_X \leq \|f\|$ for $f \in F$.*

Proof. For $n \in \mathbb{N}$ and $x_1, x_2, \dots, x_n \in X$, we can write

$$\begin{aligned} \|T(x_1, x_2, \dots, x_n, 0, 0, \dots)\|_X &= \frac{\sqrt{3}}{2} \left\| x_1 + \frac{1}{2}x_2 + \dots + \frac{1}{2^{n-1}}x_n \right\|_X \\ &\leq \frac{\sqrt{3}}{2} \left(\|x_1\|_X + \frac{1}{2}\|x_2\|_X + \dots + \frac{1}{2^{n-1}}\|x_n\|_X \right) \\ &\leq \frac{\sqrt{3}}{2} \sqrt{1 + \frac{1}{4} + \dots + \frac{1}{4^{n-1}}} \sqrt{\|x_1\|_X^2 + \|x_2\|_X^2 + \dots + \|x_n\|_X^2} \\ &= \sqrt{1 - \frac{1}{4^n}} \|(x_1, x_2, \dots, x_n, 0, 0, \dots)\|_{\ell_2(X)}. \end{aligned}$$

It follows that

$$\|Tf\|_X \leq \sqrt{1 - \frac{1}{4^d}} \|f\|_{\ell_2(X)}, \quad f \in F_d, \quad d \in \mathbb{N},$$

as the elements of F_d are supported by the first d coordinates (obviously from the definition of π).

Now, given $d \in \mathbb{N}$, we obtain for $f \in F_d$ that

$$\|Tf\|_X \leq \sqrt{1 - \frac{1}{4^d}} \|f\|_{\ell_2(X)} \leq \left(1 - \frac{1}{2^{2d+1}}\right) \|f\|_{\ell_2(X)} \leq |f|_d.$$

Therefore, the unit ball of $(F_d, |\cdot|_d)$, where $d \in \mathbb{N}$, and consequently the unit ball of $(F, \|\cdot\|)$, are subsets of $\{f \in F : \|Tf\|_X \leq 1\}$. \square

Lemma 6.7. *We have $\|Ux\| = \|x\|_X$ for $x \in X$ and the range of U is 1-complemented in $(F, \|\cdot\|)$.*

Proof. It can be easily shown that

$$TUX = x \quad \text{and} \quad \|Ux\|_{\ell_2(X)} = \|x\|_X$$

for $x \in X$. Using Lemma 6.6 and Lemma 6.5, we can write

$$\|x\|_X = \|TUX\|_X \leq \|Ux\| \leq \|Ux\|_{\ell_2(X)} = \|x\|_X, \quad x \in X.$$

Moreover, $UT : F \rightarrow F$ is a projection onto UX with $\|UT\| \leq 1$. \square

The proof of Proposition 6.2 is completed. Nevertheless, we prove here one more lemma which will be useful later.

Lemma 6.8. *We have*

$$\|f\| \geq \|P_n f\| + \frac{1}{2^{2n+4}} \|f - P_n f\|, \quad f \in F, \quad n \in \mathbb{N}.$$

Proof. Similarly as in the proof of Lemma 6.6, it is sufficient to show that the unit ball of $(F_d, |\cdot|_d)$, where $d \in \mathbb{N}$, is a subset of $\{f \in F : \|P_n f\| + \frac{1}{2^{2n+4}} \|f - P_n f\| \leq 1\}$. So, we just need to check that

$$\|P_n f\| + \frac{1}{2^{2n+4}} \|f - P_n f\| \leq |f|_d, \quad f \in F_d.$$

The inequality is clear when $d \leq n$, as $P_n f = f$. If $d \geq n + 1$, then

$$\|P_n f\| \leq |P_n f|_n \leq \left(1 - \frac{1}{2^{2n+2}}\right) \|P_n f\|_{\ell_2(X)} \leq \left(1 - \frac{1}{2^{2n+2}}\right) \|f\|_{\ell_2(X)},$$

and so

$$\begin{aligned} & \|P_n f\| + \frac{1}{2^{2n+4}} \|f - P_n f\| \\ &= \left(1 - \frac{1}{2^{2d+1}}\right) \|P_n f\| + \frac{1}{2^{2d+1}} \|P_n f\| + \frac{1}{2^{2n+4}} \|f - P_n f\| \\ &\leq \left(1 - \frac{1}{2^{2d+1}}\right) \left(1 - \frac{1}{2^{2n+2}}\right) \|f\|_{\ell_2(X)} + \frac{1}{2^{2n+3}} \|f\| + \frac{1}{2^{2n+4}} \cdot 2 \|f\| \\ &\leq \left(1 - \frac{1}{2^{2n+2}}\right) |f|_d + \frac{1}{2^{2n+3}} |f|_d + \frac{1}{2^{2n+4}} \cdot 2 |f|_d \\ &= |f|_d \end{aligned}$$

for every $f \in F_d$. □

7. RENORMING I

For the class of reflexive spaces and the class of spaces with a shrinking basis, the construction of the space F from Definition 6.4 is satisfactory. For the other two classes from Theorem 1.2, the space F needs to be renormed in a way such that the relevant isometric properties of the initial space X are preserved.

In fact, we renorm the space in two steps (renormings $\|\cdot\|_I$ and $\|\cdot\|_{II}$). For the class of non-universal spaces, one renorming is sufficient. For the class of strictly convex spaces, one more renorming is needed.

Let us accentuate two aspects of the renormings. Firstly, the new norm on the subspace $F_d = \text{span}\{f_1, \dots, f_d\}$ depends only on the old norm on F_d itself. In this way, only countably many possibilities for the norm of F_d may occur. Secondly, the norm is not changed on the subspace UX which is still an 1-complemented copy of X .

Definition 7.1. We define a seminorm

$$(24) \quad \beta(f)^2 = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{2^{4\pi^{-1}(n+1,k)}} |e_{(n,k)}^*(f) - 2e_{(n+1,k)}^*(f)|^2, \quad f \in F,$$

where $e_{(n,k)}^*$ is the system biorthogonal with the basic system $e_{(n,k)}$.

The proof of the following observation is skipped.

Fact 7.2. *For an $f \in F$, the following assertions are equivalent:*

- (i) $\beta(f) = 0$,
- (ii) $e_{(n,k)}^*(f) - 2e_{(n+1,k)}^*(f) = 0$ for all n, k ,
- (iii) $f \in UX$.

Lemma 7.3. *Let $d \in \mathbb{N} \cup \{0\}$. Then every $f \in \overline{\text{span}}\{f_{d+1}, f_{d+2}, \dots\}$ satisfies*

$$\beta(f) \leq \frac{2}{2^{2d}} \|f\|.$$

In particular, $\beta(f) \leq 2\|f\|$ for every $f \in F$.

Proof. We can compute

$$(25) \quad |e_{(n,k)}^*(f)| \leq \|e_{(n,k)}^*\|_{\ell_2(X)} \|f\|_{\ell_2(X)} \leq 2 \cdot \frac{8}{7} \|f\|,$$

$$(26) \quad |e_{(n,k)}^*(f) - 2e_{(n+1,k)}^*(f)| \leq 3 \cdot 2 \cdot \frac{8}{7} \|f\| \leq 2\sqrt{15} \|f\|.$$

Moreover, we obtain from $f \in \overline{\text{span}}\{f_{d+1}, f_{d+2}, \dots\}$ that

$$\pi^{-1}(n, k) \leq d \quad \Rightarrow \quad e_{(n,k)}^*(f) = 0,$$

and consequently

$$\pi^{-1}(n+1, k) \leq d \quad \Rightarrow \quad e_{(n,k)}^*(f) - 2e_{(n+1,k)}^*(f) = 0.$$

Therefore,

$$\begin{aligned} \beta(f)^2 &\leq \sum_{\pi^{-1}(n+1,k) > d} \frac{1}{2^{4\pi^{-1}(n+1,k)}} \cdot (2\sqrt{15}\|f\|)^2 \\ &\leq \sum_{j > d} \frac{1}{2^{4j}} \cdot 4 \cdot 15 \|f\|^2 = \frac{4}{2^{4d}} \|f\|^2, \end{aligned}$$

which proves the lemma. □

Definition 7.4. We define

$$(27) \quad \|f\|_I^2 = \|f\|^2 + \frac{1}{2^7} \beta(f)^2, \quad f \in F.$$

We note that a simple application of Lemma 7.3 gives

$$(28) \quad \|f\| \leq \|f\|_I \leq 2\|f\|.$$

Lemma 7.5. *We have $\|Ux\|_I = \|x\|_X$ for $x \in X$ and the range of U is 1-complemented in $(F, \|\cdot\|_I)$.*

Proof. Using Fact 7.2 and Lemma 6.7, we can write $\|Ux\|_I = \|Ux\| = \|x\|_X$ for $x \in X$. The projection UT works as well as in the proof of Lemma 6.7, because $\|UTf\|_I = \|UTf\| \leq \|f\| \leq \|f\|_I$ for $f \in F$. \square

Lemma 7.6. *Let $[u, v]$ be a non-degenerate line segment in F such that $\|\cdot\|_I$ is constant on $[u, v]$. Then $v - u \in UX$.*

Proof. By the same argument as in the proof of Lemma 5.4, we arrive at

$$\begin{aligned} \|u\| &= \|v\|, & \|u + v\| &= \|u\| + \|v\|, \\ \beta(u) &= \beta(v), & \beta(u + v) &= \beta(u) + \beta(v), \end{aligned}$$

and consequently

$$e_{(n,k)}^*(u) - 2e_{(n+1,k)}^*(u) = e_{(n,k)}^*(v) - 2e_{(n+1,k)}^*(v), \quad n, k \in \mathbb{N}.$$

It follows that $v - u \in UX$ by Fact 7.2. \square

Proposition 7.7. *If X is not isometrically universal for all separable Banach spaces, then $(F, \|\cdot\|_I)$ is also non-universal.*

Proof. Assume that $(F, \|\cdot\|_I)$ is isometrically universal for all separable Banach spaces. Again, let us denote

$$\Delta = \{0, 1\}^{\mathbb{N}}, \quad \Delta(i) = \{\gamma \in \Delta : \gamma(1) = i\}, \quad i = 0, 1,$$

$$Z = C(\Delta), \quad Z(i) = \{h \in Z : \gamma \notin \Delta(i) \Rightarrow h(\gamma) = 0\}, \quad i = 0, 1.$$

Considering an isometry $I : Z \rightarrow F$, we denote

$$f = I(\mathbf{1}_{\Delta(0)}).$$

We claim that the space UX (and thus the space X by Lemma 7.5) is universal, showing that I maps $Z(1)$ into UX .

Given an $h \in Z(1)$ with $\|h\| \leq 1$, we observe that $\|\mathbf{1}_{\Delta(0)}\| = \|\mathbf{1}_{\Delta(0)} \pm h\| = 1$, and so $\|f\|_I = \|f \pm Ih\|_I = 1$. By Lemma 7.6, we have $Ih \in UX$. \square

Lemma 7.8. *We have*

$$\|f\|_I \geq \|P_d f\|_I + \frac{1}{2^{2d+7}} \|f - P_d f\|_I, \quad f \in F, \quad d \in \mathbb{N}.$$

Proof. By Lemma 6.8,

$$\begin{aligned} \|f\|^2 - \|P_d f\|^2 &= (\|f\| + \|P_d f\|)(\|f\| - \|P_d f\|) \\ &\geq (\|f\| + \|P_d f\|) \cdot \frac{1}{2^{2d+4}} \|f - P_d f\|. \end{aligned}$$

At the same time, by Lemma 7.3,

$$\begin{aligned}\beta(P_d f)^2 - \beta(f)^2 &= (\beta(P_d f) + \beta(f))(\beta(P_d f) - \beta(f)) \\ &\leq (\beta(P_d f) + \beta(f)) \cdot \beta(f - P_d f) \\ &\leq 2(\|P_d f\| + \|f\|) \cdot \frac{2}{2^{2d}} \|f - P_d f\|.\end{aligned}$$

Thus, using (28), we can compute

$$\begin{aligned}\|f\|_I^2 - \|P_d f\|_I^2 &= \|f\|^2 - \|P_d f\|^2 + \frac{1}{2^7} (\beta(f)^2 - \beta(P_d f)^2) \\ &\geq \left(\frac{1}{2^{2d+4}} - \frac{1}{2^7} \cdot \frac{4}{2^{2d}} \right) (\|f\| + \|P_d f\|) \cdot \|f - P_d f\| \\ &= \frac{1}{2^{2d+5}} (\|f\| + \|P_d f\|) \cdot \|f - P_d f\| \\ &\geq \frac{1}{2^{2d+5}} \cdot \frac{1}{2} (\|f\|_I + \|P_d f\|_I) \cdot \frac{1}{2} \|f - P_d f\|_I.\end{aligned}$$

Now, it is sufficient to divide both sides by $\|f\|_I + \|P_d f\|_I$. \square

8. RENORMING II

Definition 8.1. We define a seminorm

$$(29) \quad \alpha(f)^2 = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{2^{4\pi^{-1}(n,k)}} |e_{(n,k)}^*(f)|^2, \quad f \in F,$$

where $e_{(n,k)}^*$ is the system biorthogonal with the basic system $e_{(n,k)}$.

Lemma 8.2. We have

$$\alpha(f) < \|f\|, \quad 0 \neq f \in F.$$

Proof. Using (25), we can compute

$$\alpha(f)^2 \leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{2^{4\pi^{-1}(n,k)}} \left(2 \cdot \frac{8}{7} \|f\| \right)^2 = \frac{1}{15} \left(2 \cdot \frac{8}{7} \|f\| \right)^2 < \|f\|^2,$$

which proves the lemma. \square

Fact 8.3. There is a norm ϱ on \mathbb{R}^3 such that

- $\frac{1}{2}(|r| + |s|) \leq \varrho(r, s, t) \leq \max\{|r|, |s|, |t|\}$ and, in particular, the unit sphere contains the line segment $[(1, 1, -1), (1, 1, 1)]$,
- $\varrho(r', s', t') \geq \varrho(r, s, t)$ for $0 \leq r \leq r', 0 \leq s \leq s', 0 \leq t \leq t'$,
- $\varrho(r, s, t') > \varrho(r, s, t)$ for $0 < r < s, 0 < t < t'$,
- $\varrho(r', s, t) \geq \varrho(r, s, t) + \frac{1}{4}(r' - r)$ for $r, r', s, t > 0, 0 < r < r'$.

Proof (sketch). Let a norm ϱ_0 be given by

$$B_{(\mathbb{R}^3, \varrho_0)} = \text{co} \left(\{(\pm 1, \pm 1, \pm 1)\} \cup \sqrt{2}B \right),$$

where B stands for the Euclidean unit ball of \mathbb{R}^3 . This norm satisfies the first three properties, and the norm

$$\varrho(r, s, t) = \frac{1}{4}(|r| + |s|) + \frac{1}{2}\varrho_0(r, s, t)$$

satisfies additionally the fourth one. \square

Definition 8.4. We define

$$(30) \quad \|f\|_{II} = \varrho(\|f\|, \|f\|_I, \alpha(f)), \quad f \in F.$$

We notice that a simple application of (28) and Lemma 8.2 gives

$$(31) \quad \|f\| \leq \|f\|_{II} \leq 2\|f\|,$$

since

$$\begin{aligned} \|f\| &\leq \frac{1}{2}(\|f\| + \|f\|_I) \leq \varrho(\|f\|, \|f\|_I, \alpha(f)) \\ &\leq \max \{ \|f\|, \|f\|_I, \alpha(f) \} = \|f\|_I \leq 2\|f\|. \end{aligned}$$

Lemma 8.5. We have $\|Ux\|_{II} = \|x\|_X$ for $x \in X$ and the range of U is 1-complemented in $(F, \|\cdot\|_{II})$.

Proof. Let $x \in X$ be such that $\|x\|_X = 1$. By Lemma 8.2, Lemma 6.7 and Lemma 7.5, we have $\alpha(Ux) < \|Ux\| = 1 = \|Ux\|_I$. Since the unit sphere $S_{(\mathbb{R}^3, \varrho)}$ contains the line segment $[(1, 1, -1), (1, 1, 1)]$, we obtain $\|Ux\|_{II} = 1 = \|x\|_X$. The projection UT still works, because $\|UTf\|_{II} = \|Tf\|_X = \|UTf\| \leq \|f\| \leq \|f\|_{II}$ for $f \in F$. \square

Lemma 8.6. Let $[u, v]$ be a non-degenerate line segment in F such that $\|\cdot\|_{II}$ is constant on $[u, v]$. Then u and v belong to UX .

Proof. It is enough to show that $w = \frac{1}{2}(u + v) \in UX$ (the argument can be repeated for any subsegment of $[u, v]$). Assume the opposite, i.e., $w \notin UX$. We have $\beta(w) > 0$ by Fact 7.2, and so $\|w\| < \|w\|_I$. Using the inequality

$$\alpha(w) < \frac{1}{2}(\alpha(u) + \alpha(v)),$$

a property of ϱ provides

$$\varrho\left(\|w\|, \|w\|_I, \frac{1}{2}(\alpha(u) + \alpha(v))\right) > \varrho(\|w\|, \|w\|_I, \alpha(w)) = \|w\|_{II}.$$

The computation

$$\begin{aligned} \frac{1}{2}(\|u\|_{II} + \|v\|_{II}) &= \frac{1}{2}(\varrho(\|u\|, \|u\|_I, \alpha(u)) + \varrho(\|v\|, \|v\|_I, \alpha(v))) \\ &\geq \varrho\left(\frac{1}{2}(\|u\| + \|v\|), \frac{1}{2}(\|u\|_I + \|v\|_I), \frac{1}{2}(\alpha(u) + \alpha(v))\right) \\ &\geq \varrho\left(\|w\|, \|w\|_I, \frac{1}{2}(\alpha(u) + \alpha(v))\right) > \|w\|_{II} \end{aligned}$$

concludes the proof. \square

Proposition 8.7. *If X is strictly convex, then $(F, \|\cdot\|_{II})$ is also strictly convex.*

Proof. This follows from Lemma 8.5 and Lemma 8.6. \square

Lemma 8.8. *We have*

$$\|f\|_{II} \geq \|P_d f\|_{II} + \frac{1}{2^{2d+7}} \|f - P_d f\|_{II}, \quad f \in F, d \in \mathbb{N}.$$

Proof. Using Lemma 6.8 and Lemma 7.8, we can compute

$$\begin{aligned} \|f\|_{II} &= \varrho(\|f\|, \|f\|_I, \alpha(f)) \\ &\geq \varrho\left(\|P_d f\| + \frac{1}{2^{2d+4}} \|f - P_d f\|, \|P_d f\|_I + \frac{1}{2^{2d+7}} \|f - P_d f\|_I, \alpha(f)\right) \\ &\geq \varrho\left(\|P_d f\| + \frac{1}{2^{2d+4}} \|f - P_d f\|, \|P_d f\|_I, \alpha(P_d f)\right) \\ &\geq \varrho(\|P_d f\|, \|P_d f\|_I, \alpha(P_d f)) + \frac{1}{4} \cdot \frac{1}{2^{2d+4}} \|f - P_d f\| \\ &\geq \|P_d f\|_{II} + \frac{1}{4} \cdot \frac{1}{2^{2d+4}} \cdot \frac{1}{2} \|f - P_d f\|_{II}, \end{aligned}$$

which proves the lemma. \square

9. AMALGAMATIONS OF ASPLUND AND REFLEXIVE SPACES

In the final stage of the proof of Theorem 1.2, we need some further notation. We introduce a coding of all rational Banach spaces whose basis is monotone. This enables us to provide a version of the Pełczyński universal space.

Definition 9.1. We fix a system $\{(Z_\eta, \|\cdot\|_\eta)\}_{\eta \in \mathbb{N}^{<\mathbb{N}}}$ of rational Banach spaces which satisfies the following requirements.

(a) For every η , the basis of Z_η , denoted by $z_1^\eta, z_2^\eta, \dots, z_{|\eta|}^\eta$, is monotone and consists of $|\eta|$ members in a way such that, for any two comparable sequences $\eta \subset \nu$, the space Z_ν is an extension of Z_η in the sense that the basis $z_1^\eta, z_2^\eta, \dots, z_{|\eta|}^\eta$ is 1-equivalent with $z_1^\nu, z_2^\nu, \dots, z_{|\eta|}^\nu$.

(b) Every monotone rational extension of Z_η is included as Z_ν for some $\nu \supset \eta$. More precisely, if Z is a rational space whose basis z_1, z_2, \dots, z_d is monotone and such that $z_1^\eta, z_2^\eta, \dots, z_{|\eta|}^\eta$ is 1-equivalent with $z_1, z_2, \dots, z_{|\eta|}$, then there is a $\nu \supset \eta$ with $|\nu| = d$ such that $z_1^\nu, z_2^\nu, \dots, z_{|\nu|}^\nu$ is 1-equivalent with z_1, z_2, \dots, z_d .

Definition 9.2. For every $\varphi \in \mathbb{N}^\mathbb{N}$, let $(Z_\varphi, \|\cdot\|_\varphi)$ be a Banach space with a monotone basis $z_1^\varphi, z_2^\varphi, \dots$ such that, for every $\eta \subset \varphi$, the basis $z_1^\eta, z_2^\eta, \dots, z_{|\eta|}^\eta$ of Z_η is 1-equivalent with $z_1^\varphi, z_2^\varphi, \dots, z_{|\eta|}^\varphi$.

Definition 9.3. Let U be a completion of $c_{00}(\mathbb{N}^{<\mathbb{N}} \setminus \{\emptyset\})$ with the norm defined by one of the equivalent formulae

$$(32) \quad \|x\| = \sup_{\nu \in \mathbb{N}^{<\mathbb{N}}} \left\| \sum_{\eta \subset \nu} x(\eta) z_{|\eta|}^\nu \right\|_\nu,$$

$$(33) \quad \|x\| = \sup_{\varphi \in \mathbb{N}^\mathbb{N}} \left\| \sum_{\eta \subset \varphi} x(\eta) z_{|\eta|}^\varphi \right\|_\varphi.$$

Further, let $\omega : \mathbb{N} \rightarrow \mathbb{N}^{<\mathbb{N}} \setminus \{\emptyset\}$ be a fixed non-decreasing bijection and let

$$(34) \quad u_i = \mathbf{1}_{\{\omega(i)\}}, \quad i \in \mathbb{N}.$$

As the space U is defined according to Definition 3.1, several remarkable properties follow. First of all, the sequence u_1, u_2, \dots is a monotone basis of U . If we denote

$$(35) \quad \Delta : \varphi \in \mathbb{N}^\mathbb{N} \mapsto \{\omega^{-1}((\varphi_1)) < \omega^{-1}((\varphi_1, \varphi_2)) < \dots\} \subset \mathbb{N},$$

then, using Fact 3.2, the sequences $\{z_n^\varphi : n \in \mathbb{N}\}$ and $\{u_i : i \in \Delta(\varphi)\}$ are 1-equivalent for every $\varphi \in \mathbb{N}^\mathbb{N}$. The copy $\overline{\text{span}}\{u_i : i \in \Delta(\varphi)\}$ of Z_φ is 1-complemented in U . Moreover, due to Proposition 6.2, every Banach space X with a monotone basis has an 1-complemented isometric copy in Z_φ for some $\varphi \in \mathbb{N}^\mathbb{N}$. It follows that X has an 1-complemented isometric copy also in U .

We note that the space U , including its construction and properties, is fairly similar to the space constructed and studied in [14].

Lemma 9.4. *Let \mathcal{C} be an analytic set of Banach spaces with separable dual. Then there is a $\beta < \omega_1$ such that $\text{Sz}(\ell_2(X)) \leq \beta$ for every $X \in \mathcal{C}$.*

Proof. It follows from [3, Theorem 4.11] and [3, Proposition 0.1(ii)] that $\sup\{\text{Sz}(X) : X \in \mathcal{C}'\} < \omega_1$ for any analytic set \mathcal{C}' of Banach spaces with separable dual. So, it is sufficient to find an analytic set

\mathcal{C}' which contains an isomorphic copy of $\ell_2(X)$ for every $X \in \mathcal{C}$ and every $Y \in \mathcal{C}'$ is isomorphic to $\ell_2(X)$ for some $X \in \mathcal{C}$.

Let us consider an isometry $I : \ell_2(C([0, 1])) \rightarrow C([0, 1])$ and let $\kappa : C([0, 1]) \rightarrow C([0, 1])$ be defined by $\kappa(X) = I(\ell_2(X))$ where $\ell_2(X)$ is considered as a subspace of $\ell_2(C([0, 1]))$. As κ is a Borel mapping, $\mathcal{C}' = \kappa(\mathcal{C})$ works. \square

Lemma 9.5. *For every $\beta < \omega_1$, the set*

$$(36) \quad \mathcal{A} = \{ \varphi \in \mathbb{N}^{\mathbb{N}} : \text{Sz}(Z_\varphi) \leq \beta \text{ and } z_1^\varphi, z_2^\varphi, \dots \text{ is shrinking} \}$$

is Borel in $\mathbb{N}^{\mathbb{N}}$.

Proof. By [3, Theorem 5.4(i)] and [3, Proposition 0.1(i)], the set

$$\mathcal{B} = \left\{ \{i_1 < i_2 < \dots\} \subset \mathbb{N} : \text{Sz}(\overline{\text{span}}\{u_{i_1}, u_{i_2}, \dots\}) \leq \beta \right. \\ \left. \text{and } u_{i_1}, u_{i_2}, \dots \text{ is shrinking} \right\}$$

is Borel in the space of all subsets of \mathbb{N} . As Δ is a continuous mapping, it remains to realize that $\mathcal{A} = \Delta^{-1}(\mathcal{B})$. \square

Proof of Theorem 1.2, Part 1. (Shrinking basis case.) Let \mathcal{C} be an analytic set of Banach spaces such that every member admits a monotone shrinking basis. Let $\beta < \omega_1$ be as in Lemma 9.4 and let \mathcal{A} be given by (36). By Lemma 9.5, \mathcal{A} is Borel, and thus analytic. Notice that Proposition 6.2 guarantees that every $X \in \mathcal{C}$ has an 1-complemented isometric copy in Z_φ for some $\varphi \in \mathcal{A}$.

By Lemma 2.1, there is an unrooted pruned tree T on $\mathbb{N} \times \mathbb{N}$ such that $\mathcal{A} = p[T]$ where p denotes the projection on the first coordinate. Let us consider the collection

$$(F_\sigma, \|\cdot\|_\sigma) = (Z_{p(\sigma)}, \|\cdot\|_{p(\sigma)}), \quad f_n^\sigma = z_n^{p(\sigma)}, \quad \sigma \in [T], n \in \mathbb{N}.$$

In this way, the collection $F_\sigma, \sigma \in [T]$, consists of the same spaces as the collection $Z_\varphi, \varphi \in \mathcal{A}$.

Finally, let E be the space constructed in Definition 3.1 for this collection. This space admits the required properties, due to Fact 3.2 and Proposition 3.5. \square

Lemma 9.6. *For an analytic set \mathcal{C} of Banach spaces, the set*

$$(37) \quad \mathcal{A} = \{ \varphi \in \mathbb{N}^{\mathbb{N}} : Z_\varphi \text{ is isomorphic to } \ell_2(X) \text{ for some } X \in \mathcal{C} \}$$

is analytic in $\mathbb{N}^{\mathbb{N}}$.

Proof. It is easy to show (see the proof of Lemma 9.4) that there is an analytic set \mathcal{C}' which contains an isomorphic copy of $\ell_2(X)$ for every $X \in \mathcal{C}$ and every $Y \in \mathcal{C}'$ is isomorphic to $\ell_2(X)$ for some $X \in \mathcal{C}$. By [3, Theorem 2.3(i)], the saturation

$$\begin{aligned}\mathcal{C}'' &= \{Z \in \mathcal{SE}(C([0, 1])) : Z \text{ is isomorphic to some } Y \in \mathcal{C}'\} \\ &= \{Z \in \mathcal{SE}(C([0, 1])) : Z \text{ is isomorphic to } \ell_2(X) \text{ for an } X \in \mathcal{C}\}\end{aligned}$$

is analytic.

Let $I : U \rightarrow C([0, 1])$ be an isometry. It is easy to show that the mapping

$$\zeta : \mathbb{N}^{\mathbb{N}} \rightarrow \mathcal{SE}(C([0, 1])), \quad \varphi \mapsto \overline{\text{span}}\{I(\mathbf{1}_{\{(\varphi_1)\}}), I(\mathbf{1}_{\{(\varphi_1, \varphi_2)\}}), \dots\},$$

is Borel. Due to Fact 3.2, the spaces Z_φ and $\zeta(\varphi)$ are isometric. It follows that $\mathcal{A} = \zeta^{-1}(\mathcal{C}'')$, and so that \mathcal{A} is analytic. \square

Proof of Theorem 1.2, Part 2. (Reflexive case.) Let \mathcal{C} be an analytic set of reflexive Banach spaces such that every member has a monotone basis. Let \mathcal{A} be given by (37). By Lemma 9.6, \mathcal{A} is analytic. Notice that Proposition 6.2 guarantees that every $X \in \mathcal{C}$ has an 1-complemented isometric copy in Z_φ for some $\varphi \in \mathcal{A}$. At the same time, the space Z_φ is reflexive for every $\varphi \in \mathcal{A}$.

By Lemma 2.1, there is an unrooted pruned tree T on $\mathbb{N} \times \mathbb{N}$ such that $\mathcal{A} = p[T]$ where p denotes the projection on the first coordinate. Let us consider the collection

$$(F_\sigma, \|\cdot\|_\sigma) = (Z_{p(\sigma)}, \|\cdot\|_{p(\sigma)}), \quad f_n^\sigma = z_n^{p(\sigma)}, \quad \sigma \in [T], n \in \mathbb{N}.$$

In this way, the collection $F_\sigma, \sigma \in [T]$, consists of the same spaces as the collection $Z_\varphi, \varphi \in \mathcal{A}$.

Finally, let A be the space established in Definition 4.3 for this collection. This space admits the required properties, due to Facts 4.4, 4.5 and Proposition 4.6. \square

10. AMALGAMATIONS OF NON-UNIVERSAL AND ROTUND SPACES

Definition 10.1. Let $\varphi \in \mathbb{N}^{\mathbb{N}}$ and let z_1^*, z_2^*, \dots denote the dual basic sequence of $z_1^\varphi, z_2^\varphi, \dots$. Let us define seminorms

$$(38) \quad \alpha(z)^2 = \sum_{i=1}^{\infty} \frac{1}{2^{4i}} |z_i^*(z)|^2 \quad \left(= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{2^{4\pi^{-1}(n,k)}} |z_{\pi^{-1}(n,k)}^*(z)|^2 \right),$$

$$(39) \quad \beta(z)^2 = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{2^{4\pi^{-1}(n+1,k)}} |z_{\pi^{-1}(n,k)}^*(z) - 2z_{\pi^{-1}(n+1,k)}^*(z)|^2,$$

where π is introduced in Definition 6.3. Let us further define

$$(40) \quad Z_\varphi^I = \{z \in Z_\varphi : \beta(z) < \infty\}, \quad Z_\varphi^{II} = \{z \in Z_\varphi^I : \alpha(z) < \infty\},$$

$$(41) \quad \|z\|_{\varphi,I}^2 = \|z\|_\varphi^2 + \frac{1}{2^7} \beta(z)^2, \quad z \in Z_\varphi^I,$$

$$(42) \quad \|z\|_{\varphi,II} = \varrho(\|z\|_\varphi, \|z\|_{\varphi,I}, \alpha(z)), \quad z \in Z_\varphi^{II},$$

where ϱ is a norm given by Fact 8.3.

Definition 10.2. The subspace of $(S_{C([0,1])})^\mathbb{N}$ consisting of all normalized monotone basic sequences will be denoted by \mathcal{M} .

The following proposition summarizes most of the results from Sections 6, 7 and 8.

Proposition 10.3. *There exists a Borel mapping $\Theta : \mathcal{M} \rightarrow \mathbb{N}^\mathbb{N}$ such that, for every $(e_1, e_2, \dots) \in \mathcal{M}$, if we denote $\varphi = \Theta(e_1, e_2, \dots)$ and $X = \overline{\text{span}}\{e_1, e_2, \dots\}$, then:*

(1) $Z_\varphi^I = Z_\varphi^{II} = Z_\varphi$ and the norms fulfill

$$\|z\|_\varphi \leq \|z\|_{\varphi,I} \leq 2\|z\|_\varphi, \quad \|z\|_\varphi \leq \|z\|_{\varphi,II} \leq 2\|z\|_\varphi, \quad z \in Z_\varphi.$$

(2) Both Z_φ^I and Z_φ^{II} contain an 1-complemented isometric copy of X .

(3) If X is not isometrically universal for all separable Banach spaces, then Z_φ^I is also non-universal.

(4) If X is strictly convex, then Z_φ^{II} is also strictly convex.

(5) We have

$$\|P_n z\|_{\varphi,I}^2 \geq \|P_{n-1} z\|_{\varphi,I}^2 + \left(\frac{7}{2^{2n+8}}\right)^2 |z_n^*(z)|^2, \quad z \in Z_\varphi^I, n \in \mathbb{N},$$

$$\|P_n z\|_{\varphi,II}^2 \geq \|P_{n-1} z\|_{\varphi,II}^2 + \left(\frac{7}{2^{2n+8}}\right)^2 |z_n^*(z)|^2, \quad z \in Z_\varphi^{II}, n \in \mathbb{N},$$

where z_1^*, z_2^*, \dots is the dual basic sequence and P_0, P_1, \dots is the sequence of partial sum operators associated with the basis $z_1^\varphi, z_2^\varphi, \dots$.

Proof. We realize first that the functions $l_d : \mathcal{M} \rightarrow \mathbb{N}$ from Definition 6.4 are Borel. If $l \in \mathbb{N}$, then the set of basic sequences e_1, e_2, \dots for which (18) holds with $|\cdot|_d = |\cdot|_{d,l}$ forms a closed set. Therefore, the set of sequences with $l_d = l$ is the difference of two closed sets.

Now, if a monotone basic sequence e_1, e_2, \dots is given, the properties of the system $\{(Z_\varphi, \|\cdot\|_\varphi)\}_{\varphi \in \mathbb{N}^\mathbb{N}}$ together with Lemma 6.5 guarantee that there is a $\varphi = (\varphi_1, \varphi_2, \dots)$ such that $(Z_\varphi, \|\cdot\|_\varphi)$ and $(F, \|\cdot\|)$ coincide, including their bases. To show that the choice of φ

can be Borel, it is sufficient to realize that φ can be constructed recursively in the way that φ_d depends only on l_1, \dots, l_d . This is allowed by formula (23) which implies that the norm on $\text{span}\{f_1, f_2, \dots, f_d\}$ is determined by the values l_1, \dots, l_d .

Let us check the required properties. Notice that the spaces Z_φ^I and Z_φ^{II} coincide with $(F, \|\cdot\|_I)$ and $(F, \|\cdot\|_{II})$. So, the properties easily follow from lemmata and propositions proven above.

Property (1) follows from (28) and (31) and property (2) follows from Lemma 7.5 and Lemma 8.5. Property (3) follows from Proposition 7.7 and property (4) follows from Proposition 8.7. Finally, property (5) needs a little calculation. By Lemma 6.5, we have

$$\frac{7}{8}|z_n^*(z)| \leq |z_n^*(z)| \|z_n^\varphi\|_\varphi = \|P_n z - P_{n-1} z\|_\varphi \leq \|P_n z - P_{n-1} z\|_{\varphi, I}.$$

Using Lemma 7.8, we obtain

$$\begin{aligned} \|P_n z\|_{\varphi, I}^2 &\geq \left(\|P_{n-1} z\|_{\varphi, I} + \frac{1}{2^{2(n-1)+7}} \|P_n z - P_{n-1} z\|_{\varphi, I} \right)^2 \\ &\geq \left(\|P_{n-1} z\|_{\varphi, I} + \frac{7}{2^{2n+8}} |z_n^*(z)| \right)^2 \\ &\geq \|P_{n-1} z\|_{\varphi, I}^2 + \left(\frac{7}{2^{2n+8}} \right)^2 |z_n^*(z)|^2. \end{aligned}$$

The proof of the analogous inequality for Z_φ^{II} is the same, we just use Lemma 8.8 instead of Lemma 7.8. \square

Proof of Theorem 1.2, Part 3. Suppose that \mathcal{C} is an analytic set of Banach spaces such that every member has a monotone basis. Let $\mathcal{M}_\mathcal{C}$ be the subset of \mathcal{M} consisting of all normalized monotone bases of members of \mathcal{C} . Since the mapping

$$(e_1, e_2, \dots) \mapsto \overline{\text{span}}\{e_1, e_2, \dots\}$$

is Borel, the pre-image $\mathcal{M}_\mathcal{C}$ of \mathcal{C} is analytic. Let Θ be the mapping from Proposition 10.3. The image $\Theta(\mathcal{M}_\mathcal{C})$ is an analytic subset of $\mathbb{N}^\mathbb{N}$. By Lemma 2.1, there is an unrooted pruned tree T on $\mathbb{N} \times \mathbb{N}$ such that $\Theta(\mathcal{M}_\mathcal{C}) = p[T]$ where p denotes the projection on the first coordinate. Let us consider the collections

$$(Z_\sigma^I, \|\cdot\|_{\sigma, I}) = (Z_{p(\sigma)}^I, \|\cdot\|_{p(\sigma), I}), \quad (Z_\sigma^{II}, \|\cdot\|_{\sigma, II}) = (Z_{p(\sigma)}^{II}, \|\cdot\|_{p(\sigma), II}),$$

$$f_n^\sigma = z_n^{p(\sigma)}, \quad \sigma \in [T], n \in \mathbb{N}.$$

Finally, let B^I and B^{II} be the spaces constructed in Definition 5.1 for these collections. Note that property (5) from Proposition 10.3 guarantees that the requirement (8) is fulfilled.

Both spaces B^I and B^{II} contain an 1-complemented isometric copy of every $X \in \mathcal{C}$. Indeed, a monotone basis of X is contained in $\mathcal{M}_{\mathcal{C}}$, and so property (2) from Proposition 10.3 is satisfied for some $\varphi \in \Theta(\mathcal{M}_{\mathcal{C}}) = p[T]$. If we choose a $\sigma \in [T]$ with $p(\sigma) = \varphi$, then X has an 1-complemented isometric copy in F_{σ}^I and in F_{σ}^{II} , and it is sufficient to apply Fact 5.2.

If every $X \in \mathcal{C}$ is non-universal (strictly convex), then B^I is non-universal (B^{II} is strictly convex). Indeed, in such a case, property (3) (property (4)) from Proposition 10.3 implies that the spaces Z_{φ}^I , $\varphi \in \Theta(\mathcal{M}_{\mathcal{C}})$ (Z_{φ}^{II} , $\varphi \in \Theta(\mathcal{M}_{\mathcal{C}})$), and so the spaces F_{σ}^I , $\sigma \in [T]$ (F_{σ}^{II} , $\sigma \in [T]$), are non-universal (strictly convex), and it remains just to apply Proposition 5.5. \square

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